# RadiiPolynomial

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**Abstract**

This blueprint presents a formalization in Lean 4 of the radii polynomial approach for proving existence of zeros of nonlinear functions on Banach spaces. The method combines the contraction mapping theorem with Newton-like operators, enabling computer-assisted proofs of existence and uniqueness of solutions.

The formalization covers both finite-dimensional and infinite-dimensional settings, with particular emphasis on the Banach Fixed Point Theorem (Contraction Mapping Theorem), Newton-like operators for zero-finding problems, radii polynomial approach in finite dimensions (Theorem 2.4.2), general radii polynomial approach on Banach spaces (Theorem 7.6.2), and Neumann series for operator invertibility.

The formalization extends classical results to maps between potentially different Banach spaces $E$ and $F$, requiring careful treatment of approximate inverses and approximate derivatives.

# Contents

# Notations

## General Notations

$E, F, X, Y$ — Banach spaces over $\mathbb{R}$
$\|\cdot\|$ — norm on a normed space
$\|\cdot\|_{\mathcal{B}(X,Y)}$ — operator norm for continuous linear maps
$B_r(x)$ — open ball of radius $r$ centered at $x$
$\overline{B}_r(x)$ — closed ball of radius $r$ centered at $x$
$I_E$ — identity operator on space $E$
$\mathrm{Df}(x)$ — Fréchet derivative of $f$ at $x$

## Radii Polynomial Notations

$\bar{x}$ — approximate zero or initial guess
$\tilde{x}$ — exact zero of function $f$
$A$ — approximate inverse operator
$A^\dagger$ — approximate derivative operator
$Y_0$ — bound on $\|A(f(\bar{x}))\|$ (initial defect)
$Z_0$ — bound on $\|I_E - AA^\dagger\|$ (composition error)
$Z_1$ — bound on $\|A[\mathrm{Df}(\bar{x}) - A^\dagger]\|$ (derivative approximation error)
$Z_2(r)$ — bound on $\|A[\mathrm{Df}(c) - \mathrm{Df}(\bar{x})]\|$ for $c \in \overline{B}_r(\bar{x})$
$Z(r)$ — combined bound $Z_0 + Z_1 + Z_2(r) \cdot r$
$p(r)$ — radii polynomial

## Special Cases

When $E = F$ and $A^\dagger = \mathrm{Df}(\bar{x})$, we have $Z_1 = 0$.
Simple radii polynomial: $p(r) = Z_2(r)r^2 - (1 - Z_0)r + Y_0$
General radii polynomial: $p(r) = Z_2(r)r^2 - (1 - Z_0 - Z_1)r + Y_0$

## Example 7.7: Weighted Sequence Space Notations

$\nu$ — weight parameter (positive real)
$\ell^1_\nu$ — weighted $\ell^1$ space with norm $\|a\|_{\ell^1_\nu} = \sum_{n=0}^\infty |a_n|\nu^n$
$a \star b$ — Cauchy product: $(a \star b)_n = \sum_{k=0}^n a_k b_{n-k}$
$\bar{a}$ — approximate solution (finite coefficient sequence)
$\tilde{a}$ — exact solution in $\ell^1_\nu$
$\lambda_0$ — base point for power series expansion
$c$ — parameter sequence: $c_0 = \lambda_0$, $c_1 = 1$, $c_n = 0$ for $n \geq 2$
$F(a)$ — fixed-point map: $F(a)_n = (a \star a)_n - c_n$
`toPowerSeries`$(a)$ — formal power series embedding $\ell^1_\nu \hookrightarrow \mathbb{R}[\![X]\!]$
`eval`$(a, z)$ — analytic evaluation: $\sum_{n=0}^\infty a_n z^n$ for $|z| \leq \nu$

# Chapter 1

# Foundations

## 1.1 Contraction Mapping Theorem

The contraction mapping theorem is a fundamental tool for proving existence and uniqueness of fixed points. This section establishes the basic definitions and the classical theorem.

**Definition 1.1.1** (Complete metric space)**.** *A metric space $(X, d)$ is complete if every Cauchy sequence converges in $X$, i.e., if given any $\varepsilon > 0$ there exists $N(\varepsilon)$ such that $n, m > N(\varepsilon)$ implies $d(x_n, x_m) < \varepsilon$, then there is some $y \in X$ with $\lim_{n\to\infty} x_n = y$.*

**Definition 1.1.2** (Contraction mapping)**.** *Let $(X, d)$ be a metric space. A function $T : X \to X$ is a contraction if there is a number $\kappa \in [0, 1)$, called a contraction constant, such that*

$$d(T(x), T(y)) \leq \kappa \cdot d(x, y)$$

*for all $x, y \in X$.*

**Theorem 1.1.3** (Contraction Mapping Theorem)**.** *Let $(X, d)$ be a complete metric space. If $T : X \to X$ is a contraction with contraction constant $\kappa$, then there exists a unique fixed point $\tilde{x} \in X$ of $T$. Furthermore, $\tilde{x}$ is globally attracting, and for any $x \in X$,*

$$d(T^n(x), \tilde{x}) \leq \frac{\kappa^n}{1-\kappa} d(T(x), x).$$

*Proof.* Choose $x_0 \in X$ and recursively define $x_{n+1} := T(x_n)$. By the contraction property, $d(x_{n+1}, x_n) \leq \kappa \cdot d(x_n, x_{n-1})$, so by induction $d(x_{n+1}, x_n) \leq \kappa^n d(x_1, x_0)$.

For $n < m$, the triangle inequality and geometric series give

$$d(x_n, x_m) \leq \sum_{j=n}^{m-1} d(x_{j+1}, x_j) \leq \frac{\kappa^n}{1-\kappa} d(x_1, x_0).$$

Thus $\{x_n\}$ is Cauchy. By completeness, there exists $\tilde{x} = \lim_{n\to\infty} x_n$.

By continuity of $T$, $\tilde{x} = \lim T(x_{n-1}) = T(\tilde{x})$.

For uniqueness, if $\tilde{y}$ is another fixed point with $d(\tilde{y}, \tilde{x}) > 0$, then $d(\tilde{y}, \tilde{x}) = d(T(\tilde{y}), T(\tilde{x})) \leq \kappa \cdot d(\tilde{y}, \tilde{x})$, giving $\kappa \geq 1$, a contradiction. □

**Remark 1.1.4.** *Given a contraction mapping $T : X \to X$, the rate at which points in $X$ converge to the globally attracting fixed point $\tilde{x}$ is determined by $\kappa$. In particular, the smaller $\kappa$ is, the faster iterates under $T$ converge to $\tilde{x}$.*

## 1.2 Mean Value Theorem

The mean value theorem and its corollary provide the analytical tools needed to verify contraction properties of Newton-like operators.

**Definition 1.2.1** (Operator norm). *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces and let $A : X \to Y$ be a continuous linear map. The operator norm on $A$ is given by*

$$\|A\| := \sup_{x \in X \setminus \{0\}} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{\|x\|_X = 1} \|Ax\|_Y.$$

**Proposition 1.2.2** (Properties of operator norm). *Let $(X, \|\cdot\|)$ be a normed linear space and let $A, B : X \to X$ be linear maps. Then:*
*(1) $\|Ax\| \leq \|A\|\|x\|$ for all $x \in X$*
*(2) $\|AB\| \leq \|A\|\|B\|$ (submultiplicativity)*

**Theorem 1.2.3** (Mean Value Theorem). *Suppose that $U \subset \mathbb{R}^n$ is open and that $f : U \to \mathbb{R}^n$ is $C^1$. Consider $x, y \in U$ such that the line segment $(1-t)x + ty$, $t \in [0,1]$, is contained in $U$. Then,*

$$f(y) - f(x) = \left( \int_0^1 Df((1-t)x + ty)\, dt \right) (y - x),$$

*where $Df$ denotes the Jacobian of $f$.*

*Proof.* For each $i \in \{1, \dots, n\}$, define $g_i : [0,1] \to \mathbb{R}$ by $g_i(t) = f_i((1-t)x + ty)$. By the fundamental theorem of calculus and chain rule:

$$f_i(y) - f_i(x) = g_i(1) - g_i(0) = \int_0^1 g_i'(t)\, dt = \int_0^1 \mathrm{D} f_i((1-t)x + ty) \cdot (y - x)\, dt.$$

Collecting all components yields the result. □

**Corollary 1.2.4** (Mean Value Inequality). *Consider an open set $U \subset \mathbb{R}^n$. Let $f : U \to \mathbb{R}^n$ be a $C^1$ function. Fix a point $x_0 \in U$ and assume $\overline{B}_\rho(x_0) \subset U$ for some $\rho > 0$. Then, for all $x, y \in \overline{B}_\rho(x_0)$,*

$$\|f(y) - f(x)\| \leq \left( \sup_{z \in \overline{B}_\rho(x_0)} \|Df(z)\| \right) \|y - x\|,$$

*where $\|Df(\cdot)\|$ denotes the operator norm.*

*Proof.* The line segment from $x$ to $y$ lies in $\overline{B}_\rho(x_0)$ by convexity. By the Mean Value Theorem:

$$\|f(y) - f(x)\| = \left\| \int_0^1 \mathrm{Df}((1-t)x + ty)(y - x)\, dt \right\| \leq \int_0^1 \|\mathrm{Df}((1-t)x + ty)\| \|y - x\|\, dt.$$

Since $(1-t)x + ty \in \overline{B}_\rho(x_0)$ for all $t \in [0,1]$, we have $\|\mathrm{Df}((1-t)x + ty)\| \leq \sup_{z \in \overline{B}_\rho(x_0)} \|\mathrm{Df}(z)\|$, yielding the result. □

## 1.3 Newton's Method

Newton's method transforms the problem of finding zeros into the problem of finding fixed points. This section establishes the fundamental equivalence and introduces Newton-like operators.

**Definition 1.3.1** (Newton-like map)**.** *Let $E, F$ be Banach spaces, $f : E \to F$ a function, and $A : F \to E$ a continuous linear map. The Newton-like map is defined by*

$$T(x) = x - A(f(x)).$$

**Proposition 1.3.2** (Fixed points $\iff$ Zeros (Proposition 2.3.1))**.** *Let $E, F$ be vector spaces, $f : E \to F$, and $A : F \to E$ an injective linear map. Let $T(x) = x - A(f(x))$ be the Newton-like operator. Then:*

$$T(x) = x \quad \iff \quad f(x) = 0.$$

*Proof.* ($\Rightarrow$) If $T(x) = x$, then $x - A(f(x)) = x$, so $A(f(x)) = 0$. Since $A$ is linear, $A(0) = 0$. By injectivity of $A$, $A(f(x)) = A(0)$ implies $f(x) = 0$.

($\Leftarrow$) If $f(x) = 0$, then $T(x) = x - A(0) = x - 0 = x$. □

## 1.4 Radii Polynomials in Finite Dimensions

This section develops the radii polynomial method for proving existence of zeros in finite dimensions. The method provides explicit domains of existence and uniqueness.

**Theorem 1.4.1** (Fixed point theorem with radii polynomial (Theorem 2.4.1))**.** *Consider a map $T \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and let $\bar{x} \in \mathbb{R}^n$. Let $Y_0 \geq 0$ and $Z : (0, \infty) \to [0, \infty)$ be a non-negative function satisfying*

$$\|T(\bar{x}) - \bar{x}\| \leq Y_0$$

$$\|DT(c)\| \leq Z(r), \quad \text{for all } c \in \overline{B}_r(\bar{x}) \text{ and all } r > 0.$$

*Define*

$$p(r) := (Z(r) - 1)r + Y_0.$$

*If there exists $r_0 > 0$ such that $p(r_0) < 0$, then there exists a unique $\tilde{x} \in \overline{B}_{r_0}(\bar{x})$ such that $T(\tilde{x}) = \tilde{x}$.*

*Proof.* The assumption $p(r_0) < 0$ implies $Z(r_0)r_0 + Y_0 < r_0$ and hence $Z(r_0) < 1$.

**Step 1: $T$ maps $\overline{B}_{r_0}(\bar{x})$ into itself.** Let $x \in \overline{B}_{r_0}(\bar{x})$. By the Mean Value Inequality:

$$\|T(x) - \bar{x}\| \leq \|T(x) - T(\bar{x})\| + \|T(\bar{x}) - \bar{x}\| \leq Z(r_0)\|x - \bar{x}\| + Y_0 \leq Z(r_0)r_0 + Y_0 < r_0.$$

**Step 2: $T$ is a contraction on $\overline{B}_{r_0}(\bar{x})$.** For $a, b \in \overline{B}_{r_0}(\bar{x})$, by the Mean Value Inequality: $\|T(a) - T(b)\| \leq Z(r_0)\|a - b\|$. Since $Z(r_0) < 1$, $T$ is a contraction.

**Step 3: Apply Contraction Mapping Theorem.** By Theorem 1.1.3, there exists a unique $\tilde{x} \in \overline{B}_{r_0}(\bar{x})$ with $T(\tilde{x}) = \tilde{x}$. □

**Definition 1.4.2** (Radii polynomial)**.** *Given constants $Y_0, Z_0 \geq 0$ and a function $Z_2 : (0, \infty) \to [0, \infty)$, the radii polynomial is defined by*

$$p(r) := Z_2(r)r^2 - (1 - Z_0)r + Y_0.$$

**Definition 1.4.3** (Combined bound)**.** *The combined bound is defined as*

$$Z(r) := Z_0 + Z_2(r) \cdot r.$$

**Lemma 1.4.4** (Alternative form of radii polynomial)**.** *The radii polynomial can be rewritten as*

$$p(r) = (Z(r) - 1)r + Y_0.$$

*Proof.* Direct calculation: $(Z_0 + Z_2(r)r - 1)r + Y_0 = Z_2(r)r^2 + Z_0 r - r + Y_0 = Z_2(r)r^2 - (1 - Z_0)r + Y_0$. ☐

**Lemma 1.4.5** (Polynomial negativity implies contraction)**.** *If $Y_0 \geq 0$, $r_0 > 0$, and $p(r_0) < 0$, then $Z(r_0) < 1$.*

*Proof.* By Lemma 1.4.4, $p(r_0) = (Z(r_0) - 1)r_0 + Y_0 < 0$. Since $Y_0 \geq 0$, we have $(Z(r_0) - 1)r_0 < 0$. Since $r_0 > 0$, dividing gives $Z(r_0) - 1 < 0$, i.e., $Z(r_0) < 1$. ☐

The main radii polynomial theorem (Theorem 2.4.2) is stated and proved in full generality for Banach spaces in Section 2.2 as Theorem 2.2.23. Here we apply it to the finite-dimensional case $E = \mathbb{R}^n$.

### 1.4.1 Example 2.4.5: Finding $\sqrt{2}$

We demonstrate the radii polynomial method on the simplest nonlinear function $f(x) = x^2 - 2$, verifying the existence of a unique zero near $\bar{x} = 1.3$.

**Theorem 1.4.6** (Square root verification (Example 2.4.5))**.** *Consider $f(x) = x^2 - 2$. Choose initial guess $\bar{x} = \frac{13}{10} = 1.3$, approximate inverse $A = \frac{19}{50} = 0.38 \approx (f'(\bar{x}))^{-1} = (2\bar{x})^{-1}$, bounds $Y_0 = \frac{3}{25} = 0.12$, $Z_0 = \frac{3}{250} = 0.012$, $Z_2 = \frac{19}{25} = 0.76$, and radius $r_0 = \frac{3}{20} = 0.15$.*

*The radii polynomial $p(r) = 0.76r^2 - 0.988r + 0.12$ satisfies $p(0.15) < 0$.*

*Therefore, there exists a unique $\tilde{x} \in \overline{B}_{0.15}(1.3) = [1.15, 1.45]$ with $\tilde{x}^2 = 2$ and $f'(\tilde{x}) = 2\tilde{x}$ invertible.*

*Proof.* **Step 1: Compute bounds.** For the $Y_0$ bound: $|Af(\bar{x})| = |0.38(1.3^2 - 2)| = |0.38 \cdot (-0.31)| = 0.1178 \leq 0.12 = Y_0$.

For the $Z_0$ bound: $|1 - A \cdot 2\bar{x}| = |1 - 0.38 \cdot 2.6| = |1 - 0.988| = 0.012 = Z_0$.

For the $Z_2$ bound: If $c \in \overline{B}_r(\bar{x})$, then $|A[f'(c) - f'(\bar{x})]| = |A(2c - 2\bar{x})| = 2|A||c - \bar{x}| \leq 2 \cdot 0.38 \cdot r = 0.76r = Z_2 r$.

**Step 2: Verify polynomial negativity.** $p(0.15) = 0.76 \cdot 0.15^2 - 0.988 \cdot 0.15 + 0.12 = 0.0171 - 0.1482 + 0.12 = -0.0111 < 0$.

**Step 3: Apply Theorem 2.2.23.** The theorem guarantees a unique zero $\tilde{x} \in [1.15, 1.45]$ with invertible derivative. Since $\sqrt{2} \approx 1.414 \in [1.15, 1.45]$, this zero is $\sqrt{2}$. ☐

**Corollary 1.4.7.** *There exists a unique $\tilde{x} \in \overline{B}_{3/20}(13/10)$ with $\tilde{x}^2 = 2$.*

**Remark 1.4.8.** ***Optimal choice:*** *If $\bar{x} = \sqrt{2}$ (exact) and $A = (2\bar{x})^{-1}$ (exact inverse), the radii polynomial becomes $p(r) = \frac{\sqrt{2}}{2}r^2 - r$, giving $EI(p) = (0, \sqrt{2})$.*

***Approximate choice:*** *With $\bar{x} = 1.3$ and $A = 0.38$, the existence interval is approximately $EI(p) \approx (0.136, 1.164)$, still sufficient to verify the zero.*

# Chapter 2

# Radii Polynomial on Banach Spaces

## 2.1 Weighted $\ell^1_\nu$ Banach Algebra (Section 7.4)

This section establishes the weighted $\ell^1_\nu$ space as a commutative Banach algebra under the Cauchy product. The algebraic structure is derived by connecting to `PowerSeries` multiplication, then the analytic properties (submultiplicativity) are proven using weight factorization.

### 2.1.1 Positive Reals and Scaled Spaces

**Definition 2.1.1** (Positive real numbers). *The type of positive real numbers:* `PosReal` $:= \{x : \mathbb{R} \mid 0 < x\}$.

**Definition 2.1.2** (Scaled real type). *For $\nu > 0$ and $n \in \mathbb{N}$, the type* `ScaledReal` $\nu$ $n$ *is* $\mathbb{R}$ *equipped with the scaled norm* $\|x\|_n := |x| \cdot \nu^n$.

### 2.1.2 Weighted $\ell^p$ Spaces

**Definition 2.1.3** (Weighted $\ell^p$ space). *The weighted $\ell^p_\nu$ space is realized as* `lp` (`ScaledReal` $\nu$) $p$. *The norm is:*

$$\|a\|_{p,\nu} := \left( \sum_{n=0}^{\infty} |a_n|^p \nu^{pn} \right)^{1/p}.$$

**Definition 2.1.4** (Weighted $\ell^1$ space). *Specialization of $\ell^p_\nu$ to $p = 1$:*

$$\ell^1_\nu = \left\{ a : \mathbb{N} \to \mathbb{R} \;\middle|\; \|a\|_{1,\nu} := \sum_{n=0}^{\infty} |a_n| \nu^n < \infty \right\}.$$

**Lemma 2.1.5** (Completeness). *The space $\ell^p_\nu$ is complete (a Banach space) for $p \geq 1$.*

**Lemma 2.1.6** (Membership criterion). *A sequence $a$ belongs to $\ell^1_\nu$ iff $\sum_{n=0}^{\infty} |a_n| \nu^n$ converges.*

**Lemma 2.1.7** (Weight factorization). *For $k + l = n$: $\nu^k \cdot \nu^l = \nu^n$. This is the key property enabling submultiplicativity.*

### 2.1.3 Cauchy Product (Definition 7.4.2)

The Cauchy product (convolution) defines multiplication on sequence spaces.

**Definition 2.1.8** (Cauchy product). *The* Cauchy product *of sequences* $a, b : \mathbb{N} \to R$ *is:*

$$(a \star b)_n = \sum_{k+l=n} a_k b_l = \sum_{j=0}^{n} a_{n-j} b_j.$$

**Theorem 2.1.9** (Connection to PowerSeries). *Let* $\texttt{toPowerSeries}(a) = \sum_n a_n X^n$*. Then:*

$$\texttt{toPowerSeries}(a \star b) = \texttt{toPowerSeries}(a) \cdot \texttt{toPowerSeries}(b).$$

*This bridge lets us transport all ring axioms from* `PowerSeries R`*.*

**Theorem 2.1.10** (Associativity). $(a \star b) \star c = a \star (b \star c)$.

**Theorem 2.1.11** (Commutativity). $a \star b = b \star a$ *(when* $R$ *is commutative).*

**Theorem 2.1.12** (Left distributivity). $a \star (b + c) = a \star b + a \star c$.

**Theorem 2.1.13** (Right distributivity). $(a + b) \star c = a \star c + b \star c$.

**Definition 2.1.14** (Identity sequence). *The identity element is the Kronecker delta:*

$$e_n = \delta_{n,0} = \begin{cases} 1 & n = 0 \\ 0 & n \geq 1. \end{cases}$$

**Theorem 2.1.15** (Left identity). $e \star a = a$.

**Theorem 2.1.16** (Right identity). $a \star e = a$.

### 2.1.4 Scalar-Sequence Compatibility

These lemmas establish compatibility between scalar multiplication and the Cauchy product. They are essential for the Fréchet derivative formula in Section 2.3.

**Theorem 2.1.17** (Left scalar multiplication). *Scalars pull out on the left:* $(c \cdot a) \star b = c \cdot (a \star b)$.

**Theorem 2.1.18** (Right scalar multiplication). *Scalars pull out on the right:* $a \star (c \cdot b) = c \cdot (a \star b)$. *Requires commutativity of the coefficient ring.*

### 2.1.5 Banach Algebra Structure (Theorem 7.4.4)

**Lemma 2.1.19** (Closure under multiplication). *If* $a, b \in \ell^1_\nu$*, then* $a \star b \in \ell^1_\nu$.

*Proof.* By Mertens' theorem and weight factorization:

$$\begin{aligned} \sum_{n=0}^{\infty} |(a \star b)_n| \nu^n &\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_k||b_{n-k}| \nu^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_k| \nu^k \cdot |b_{n-k}| \nu^{n-k} \\ &= \|a\|_{\ell^1_\nu} \cdot \|b\|_{\ell^1_\nu} < \infty. \end{aligned}$$

□

**Definition 2.1.20** (Multiplication on $\ell^1_\nu$). $\texttt{mul}(a, b) := a \star b$ *lifted to* $\ell^1_\nu$.

**Theorem 2.1.21** (Submultiplicativity). $\|a \star b\|_{1,\nu} \leq \|a\|_{1,\nu} \cdot \|b\|_{1,\nu}$.

*This is **axiom (4)** of Definition 7.4.1, the key analytic property. The proof uses Mertens' theorem and weight factorization* $\nu^n = \nu^k \cdot \nu^l$.

**Definition 2.1.22** (Identity element). *The identity* $e \in \ell^1_\nu$ *with* $e_0 = 1$, $e_n = 0$ *for* $n \geq 1$.

**Lemma 2.1.23** (Norm of identity). $\|e\|_{1,\nu} = 1$.

### 2.1.6 Typeclass Instances (Corollary 7.4.5)

**Theorem 2.1.24** (Ring instance). $\ell^1_\nu$ *is a ring under Cauchy product multiplication.*

**Theorem 2.1.25** (Commutative ring instance). $\ell^1_\nu$ *is a commutative ring.*

**Theorem 2.1.26** (Normed ring instance). $\ell^1_\nu$ *is a normed ring (submultiplicativity holds).*

**Theorem 2.1.27** (Norm one class). $\|1\| = 1$ *in* $\ell^1_\nu$.

### 2.1.7 Algebra Structure

These instances establish that $\ell^1_\nu$ is an $\mathbb{R}$-algebra, meaning scalar multiplication is compatible with ring multiplication. This is essential for the Fréchet derivative formula $D[\text{sq}](a)h = 2(a \star h)$, where the scalar 2 must commute with the multiplication.

**Theorem 2.1.28** (Scalar commutes with multiplication). *For* $c \in \mathbb{R}$ *and* $a, b \in \ell^1_\nu$*:*

$$c \cdot (a \star b) = a \star (c \cdot b).$$

*This is the* `SMulCommClass` *instance.*

**Theorem 2.1.29** (Scalar tower property). *For* $c \in \mathbb{R}$ *and* $a, b \in \ell^1_\nu$*:*

$$(c \cdot a) \star b = c \cdot (a \star b).$$

*This is the* `IsScalarTower` *instance.*

**Theorem 2.1.30** ($\mathbb{R}$-Algebra instance). $\ell^1_\nu$ *is an* $\mathbb{R}$*-algebra. The algebra map* $\mathbb{R} \to \ell^1_\nu$ *sends* $r$ *to* $r \cdot e$ *where* $e$ *is the identity sequence.*

**Lemma 2.1.31** (Norm of scalar multiplication). *For* $c \in \mathbb{R}$ *and* $a \in \ell^1_\nu$*:*

$$\|c \cdot a\|_{1,\nu} = |c| \cdot \|a\|_{1,\nu}.$$

*Proof.* Direct computation: $\|c \cdot a\| = \sum_n |c \cdot a_n| \nu^n = |c| \sum_n |a_n| \nu^n = |c| \cdot \|a\|$. □

**Theorem 2.1.32** (Normed $\mathbb{R}$-Algebra instance). $\ell^1_\nu$ *is a normed* $\mathbb{R}$*-algebra, satisfying* $\|c \cdot a\| \leq |c| \cdot \|a\|$*. (In fact, equality holds.)*

**Lemma 2.1.33** (Norm bound for powers). *For* $a \in \ell^1_\nu$ *and* $n \in \mathbb{N}$*:*

$$\|a^n\|_{1,\nu} \leq \|a\|^n_{1,\nu}.$$

*Proof.* By induction on $n$. The base case $n = 0$ gives $\|1\| = 1 = \|a\|^0$. For the inductive step, submultiplicativity gives $\|a^{n+1}\| = \|a \cdot a^n\| \leq \|a\| \cdot \|a^n\| \leq \|a\| \cdot \|a\|^n = \|a\|^{n+1}$. □

**Corollary 2.1.34** (Commutative Banach algebra). *$\ell^1_\nu$ is a commutative Banach algebra over $\mathbb{R}$:*

1. *Complete normed space (Banach)*
2. *Commutative ring under $\star$*
3. *$\mathbb{R}$-algebra with compatible scalar multiplication*
4. *Submultiplicative: $\|a \star b\| \leq \|a\| \cdot \|b\|$*
5. *Scalar-norm compatibility: $\|c \cdot a\| = |c| \cdot \|a\|$*
6. *$\|1\| = 1$*

*This is the full Mathlib `NormedAlgebra` + `CompleteSpace` structure.*

### 2.1.8 Architecture Summary

The formalization separates concerns into layers:

| File | Contents | Dependencies |
|---|---|---|
| `CauchyProduct.lean` | Ring axioms, scalar compat | `PowerSeries.Basic` |
| `lpWeighted.lean` | Banach structure, instances | `CauchyProduct`, `lpSpace` |
| `FrechetCauchyProduct.lean` | Fréchet derivatives | Ring instances |

Key insight: Ring axioms are transported from `PowerSeries` via `toPowerSeries_mul`, avoiding manual verification of associativity/distributivity. The `SMulCommClass` and `IsScalarTower` instances similarly lift scalar compatibility from `CauchyProduct`.

## 2.2 Radii Polynomial Approach on Banach Spaces

This section extends the radii polynomial method to infinite-dimensional Banach spaces, allowing for maps between potentially different spaces $E$ and $F$.

### 2.2.1 Banach Space Setup

We work with two Banach spaces $E$ and $F$ over $\mathbb{R}$. For each space $X \in \{E, F\}$:

(1) `NormedAddCommGroup X`: $X$ has a norm satisfying definiteness, symmetry, triangle inequality

(2) `NormedSpace ℝ X`: The norm is compatible with scalar multiplication

(3) `CompleteSpace X`: Every Cauchy sequence converges (crucial for fixed point theorems)

This framework supports:

(4) Fréchet derivatives (via the norm structure)

(5) Fixed point theorems (via completeness)

(6) Mean Value Theorem (via the metric structure)

(7) Linear operator theory (via the vector space structure)

### 2.2.2 Neumann Series and Operator Invertibility

The Neumann series provides a constructive way to show operators close to the identity are invertible.

**Theorem 2.2.1** (Neumann series invertibility). *Let $E$ be a Banach space and $B : E \to E$ a continuous linear operator. If $\|I_E - B\| < 1$, then $B$ is invertible (a unit in the multiplicative sense).*

*Proof.* Write $B = I_E - (I_E - B)$. Since $\|I_E - B\| < 1$, the Neumann series $\sum_{n=0}^{\infty}(I_E - B)^n$ converges absolutely to $(I_E - (I_E - B))^{-1} = B^{-1}$. This is a direct application of Mathlib's `isUnit_one_sub_of_norm_lt_one`. □

**Lemma 2.2.2** (Explicit two-sided inverse). *If $\|I_E - B\| < 1$ for $B : E \to E$, then there exists $B^{-1} : E \to E$ such that*

$$B \circ B^{-1} = I_E \quad \text{and} \quad B^{-1} \circ B = I_E.$$

**Lemma 2.2.3** (Composition form). *If $\|I_E - B\| < 1$, then there exists $B^{-1}$ such that*

$$B.comp(B^{-1}) = I_E \quad \text{and} \quad B^{-1}.comp(B) = I_E.$$

### 2.2.3 Newton-Like Operators for E to F Maps

**Definition 2.2.4** (Newton-like map for E to F). *For a function $f : E \to F$ and an approximate inverse $A : F \to E$, the Newton-like map is*

$$T(x) = x - A(f(x)).$$

*Note that $T : E \to E$ even though $f$ maps between different spaces.*

**Proposition 2.2.5** (Fixed points ⇔ Zeros for E to F). *Let $f : E \to F$ and $A : F \to E$ be injective. Then for the Newton-like operator $T(x) = x - A(f(x))$:*

$$T(x) = x \quad \Longleftrightarrow \quad f(x) = 0.$$

*This holds even when $E \neq F$; injectivity of $A$ is sufficient.*

### 2.2.4 Radii Polynomial Definitions

**Definition 2.2.6** (General radii polynomial). *For constants $Y_0, Z_0, Z_1 \geq 0$ and function $Z_2 : (0, \infty) \to [0, \infty)$, the general radii polynomial is*

$$p(r) := Z_2(r)r^2 - (1 - Z_0 - Z_1)r + Y_0.$$

**Definition 2.2.7** (Combined bound (general case)). *The combined bound is*

$$Z(r) := Z_0 + Z_1 + Z_2(r) \cdot r.$$

**Lemma 2.2.8** (Alternative form (general)). *The general radii polynomial can be rewritten as $p(r) = (Z(r) - 1)r + Y_0$.*

**Lemma 2.2.9** (General polynomial negativity implies contraction). *If $Y_0 \geq 0$, $r_0 > 0$, and $p(r_0) < 0$ for the general radii polynomial, then $Z(r_0) < 1$.*

**Definition 2.2.10** (Simple radii polynomial). *For $Y_0 \geq 0$ and $Z : (0, \infty) \to [0, \infty)$, the simple radii polynomial is*

$$p(r) := (Z(r) - 1)r + Y_0.$$

**Lemma 2.2.11** (Simple polynomial negativity implies contraction). *If $Y_0 \geq 0$, $r_0 > 0$, and $p(r_0) < 0$ for the simple radii polynomial, then $Z(r_0) < 1$.*

### 2.2.5 Operator Bounds

**Lemma 2.2.12** ($Y_0$ bound for Newton operator)**.** *If $\|A(f(\bar{x}))\| \leq Y_0$, then for $T(x) = x - A(f(x))$: $\|T(\bar{x}) - \bar{x}\| \leq Y_0$.*

*Proof.* $\|T(\bar{x}) - \bar{x}\| = \|(\bar{x} - A(f(\bar{x}))) - \bar{x}\| = \|-A(f(\bar{x}))\| = \|A(f(\bar{x}))\| \leq Y_0$. □

**Lemma 2.2.13** (Derivative of Newton operator)**.** *For $T(x) = x - A(f(x))$ where $f : E \to F$ is differentiable: $DT(x) = I_E - A \circ Df(x)$.*

*Proof.* By the chain rule: $\mathrm{D}[x \mapsto A(f(x))] = A \circ \mathrm{Df}(x)$. Since $\mathrm{D}[\mathrm{id}] = I_E$, we have $\mathrm{DT}(x) = I_E - A \circ \mathrm{Df}(x)$. □

**Lemma 2.2.14** (General derivative bound)**.** *Suppose for all $c \in \overline{B}_r(\bar{x})$:*

$$\|I_E - A \circ A^\dagger\| \leq Z_0$$

$$\|A \circ (A^\dagger - Df(\bar{x}))\| \leq Z_1$$

$$\|A \circ (Df(c) - Df(\bar{x}))\| \leq Z_2(r) \cdot r$$

*Then $\|DT(c)\| \leq Z_0 + Z_1 + Z_2(r) \cdot r = Z(r)$.*

*Proof.* Decompose using $A^\dagger$:

$$I_E - A \circ \mathrm{Df}(c) = [I_E - A \circ A^\dagger] + A \circ [A^\dagger - \mathrm{Df}(\bar{x})] + A \circ [\mathrm{Df}(\bar{x}) - \mathrm{Df}(c)].$$

By triangle inequality: $\|\mathrm{DT}(c)\| \leq Z_0 + Z_1 + Z_2(r) \cdot r$. □

**Lemma 2.2.15** (Simple derivative bound)**.** *When $A^\dagger = Df(\bar{x})$ (so $Z_1 = 0$), for all $c \in \overline{B}_r(\bar{x})$: if $\|I_E - A \circ Df(\bar{x})\| \leq Z_0$ and $\|A \circ (Df(c) - Df(\bar{x}))\| \leq Z_2(r) \cdot r$, then $\|DT(c)\| \leq Z_0 + Z_2(r) \cdot r$.*

### 2.2.6 Helper Lemmas

**Lemma 2.2.16** (Closed balls are complete)**.** *In a complete space $E$, closed balls $\overline{B}_r(x)$ are complete.*

**Lemma 2.2.17** (Extended distance is finite)**.** *In normed spaces, extended distance is always finite: $d_{ext}(x, y) \neq \top$.*

**Lemma 2.2.18** (Constructing derivative inverse)**.** *If $A : F \to E$ is injective and $\|I_E - A \circ B\| < 1$ for $B : E \to F$, then $B$ is invertible with inverse $B^{-1} = (A \circ B)^{-1} \circ A$.*

*Proof.* By Lemma 2.2.3, $A \circ B$ is invertible with inverse $(A \circ B)^{-1}$. Let $B^{-1} = (A \circ B)^{-1} \circ A$.

**Left inverse:** $B(B^{-1}(x)) = B((A \circ B)^{-1}(A(x)))$. Apply $A$: $A(B(B^{-1}(x))) = (A \circ B)((A \circ B)^{-1}(A(x))) = A(x)$. By injectivity of $A$: $B(B^{-1}(x)) = x$.

**Right inverse:** $B^{-1}(B(x)) = (A \circ B)^{-1}(A(B(x))) = (A \circ B)^{-1}((A \circ B)(x)) = x$. □

**Lemma 2.2.19** (Ball mapping property)**.** *Given $T : E \to E$ differentiable with: (a) $\|T(\bar{x}) - \bar{x}\| \leq Y_0$; (b) $\|DT(c)\| \leq Z(r_0)$ for all $c \in \overline{B}_{r_0}(\bar{x})$; (c) $Z(r_0) \geq 0$; (d) $p(r_0) = (Z(r_0) - 1)r_0 + Y_0 < 0$. Then $T : \overline{B}_{r_0}(\bar{x}) \to \overline{B}_{r_0}(\bar{x})$.*

*Proof.* From $p(r_0) < 0$: $Z(r_0) \cdot r_0 + Y_0 < r_0$. The segment $[\bar{x}, x]$ lies in $\overline{B}_{r_0}(\bar{x})$ by convexity. By Mean Value Theorem: $\|T(x) - T(\bar{x})\| \leq Z(r_0) \cdot \|x - \bar{x}\| \leq Z(r_0) \cdot r_0$. By triangle inequality:

$$\|T(x) - \bar{x}\| \leq \|T(x) - T(\bar{x})\| + \|T(\bar{x}) - \bar{x}\| \leq Z(r_0) \cdot r_0 + Y_0 < r_0.$$

□

### 2.2.7 Main Theorems

**Theorem 2.2.20** (General Fixed Point Theorem (Theorem 7.6.1)). *Let $T : E \to E$ be Fréchet differentiable and $\bar{x} \in E$. Suppose:*

$$\|T(\bar{x}) - \bar{x}\| \leq Y_0$$

$$\|DT(x)\| \leq Z(r) \quad \text{for all } x \in \overline{B}_r(\bar{x})$$

*Define $p(r) := (Z(r) - 1)r + Y_0$.*

*If there exists $r_0 > 0$ such that $p(r_0) < 0$, then there exists a unique $\tilde{x} \in \overline{B}_{r_0}(\bar{x})$ such that $T(\tilde{x}) = \tilde{x}$.*

*Proof.* **Step 1:** From $p(r_0) < 0$ and Lemma 2.2.11: $Z(r_0) < 1$. Also $Z(r_0) \geq 0$ since $\|\mathrm{DT}(\bar{x})\| \geq 0$.

**Step 2:** By Lemma 2.2.19, $T : \overline{B}_{r_0}(\bar{x}) \to \overline{B}_{r_0}(\bar{x})$.

**Step 3:** $T$ restricted to $\overline{B}_{r_0}(\bar{x})$ is a contraction with constant $Z(r_0) < 1$. For $x, y \in \overline{B}_{r_0}(\bar{x})$, by Mean Value Theorem: $\|T(x) - T(y)\| \leq Z(r_0)\|x - y\|$.

**Step 4:** The closed ball is complete by Lemma 2.2.16.

**Step 5:** Apply Theorem 1.1.3 to get unique fixed point. □

**Theorem 2.2.21** (General Radii Polynomial Theorem (Theorem 7.6.2)). *Let $E$ and $F$ be Banach spaces and $f : E \to F$ be Fréchet differentiable. Suppose $\bar{x} \in E$, $A^\dagger : E \to F$, and $A : F \to E$ with $A$ injective. Assume:*

$$\|A(f(\bar{x}))\| \leq Y_0$$

$$\|I_E - A \circ A^\dagger\| \leq Z_0$$

$$\|A \circ [Df(\bar{x}) - A^\dagger]\| \leq Z_1$$

$$\|A \circ [Df(c) - Df(\bar{x})]\| \leq Z_2(r) \cdot r \quad \text{for } c \in \overline{B}_r(\bar{x})$$

*Define $p(r) := Z_2(r)r^2 - (1 - Z_0 - Z_1)r + Y_0$.*

*If there exists $r_0 > 0$ such that $p(r_0) < 0$, then there exists a unique $\tilde{x} \in \overline{B}_{r_0}(\bar{x})$ with $f(\tilde{x}) = 0$.*

*Proof.* **Step 1:** Let $T(x) = x - A(f(x))$. Then $T : E \to E$ is differentiable.

**Step 2:** By Lemma 2.2.12: $\|T(\bar{x}) - \bar{x}\| \leq Y_0$. By Lemma 2.2.14: $\|\mathrm{DT}(c)\| \leq Z(r_0)$ for $c \in \overline{B}_{r_0}(\bar{x})$. By Lemma 2.2.8: $p(r_0) = (Z(r_0) - 1)r_0 + Y_0 < 0$.

**Step 3:** Apply Theorem 2.2.20: unique $\tilde{x} \in \overline{B}_{r_0}(\bar{x})$ with $T(\tilde{x}) = \tilde{x}$.

**Step 4:** By Proposition 2.2.5 with injectivity of $A$: $f(\tilde{x}) = 0$. □

**Theorem 2.2.22** (Simple Radii Polynomial for E to F). *Given $f : E \to F$ Fréchet differentiable and injective $A : F \to E$ satisfying $\|A(f(\bar{x}))\| \leq Y_0$, $\|I_E - A \circ Df(\bar{x})\| \leq Z_0$, and $\|A \circ [Df(c) - Df(\bar{x})]\| \leq Z_2(r) \cdot r$ for $c \in \overline{B}_r(\bar{x})$. If $p(r_0) = Z_2(r_0)r_0^2 - (1 - Z_0)r_0 + Y_0 < 0$, then there exists unique $\tilde{x} \in \overline{B}_{r_0}(\bar{x})$ with $f(\tilde{x}) = 0$.*

**Theorem 2.2.23** (Simple Radii Polynomial (Same Space)). *Consider $f : E \to E$ Fréchet differentiable, $\bar{x} \in E$, and $A : E \to E$ injective. Assume $\|A(f(\bar{x}))\| \leq Y_0$, $\|I_E - A \circ Df(\bar{x})\| \leq Z_0$, and $\|A \circ [Df(c) - Df(\bar{x})]\| \leq Z_2(r) \cdot r$ for $c \in \overline{B}_r(\bar{x})$. Define $p(r) := Z_2(r)r^2 - (1 - Z_0)r + Y_0$.*

*If $p(r_0) < 0$, then there exists unique $\tilde{x} \in \overline{B}_{r_0}(\bar{x})$ with $f(\tilde{x}) = 0$ and $Df(\tilde{x})$ invertible.*

*Proof.* **Steps 1–4:** As in Theorem 2.2.21, get fixed point $\tilde{x}$ with $f(\tilde{x}) = 0$.

**Step 5 (Invertibility):** Since $\tilde{x} \in \overline{B}_{r_0}(\bar{x})$ and $Z(r_0) < 1$ (by Lemma 1.4.5), $\|I_E - A \circ \mathrm{Df}(\tilde{x})\| \leq Z(r_0) < 1$. Apply Lemma 2.2.18: $\mathrm{Df}(\tilde{x})$ is invertible. □

## 2.3 Example 7.7: Square Root via Power Series

This section applies the radii polynomial method to prove existence of an analytic branch of $\sqrt{\lambda}$ near a base point $\lambda_0 > 0$. We work in the weighted sequence space $\ell^1_\nu$ (Section 2.1) and verify all bounds to establish existence of a fixed point, then interpret this fixed point as a convergent power series.

### 2.3.1 The Fixed-Point Problem

We seek an analytic function $x(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda - \lambda_0)^n$ satisfying $x(\lambda)^2 = \lambda$. In coefficient space, this becomes: find $a \in \ell^1_\nu$ such that $(a \star a)_n = c_n$ where $c$ is the parameter sequence.

**Definition 2.3.1** (Parameter sequence). *For $\lambda_0 \in \mathbb{R}$, the parameter sequence is*

$$c_n = \begin{cases} \lambda_0 & n = 0 \\ 1 & n = 1 \\ 0 & n \geq 2 \end{cases}$$

*This encodes the polynomial $\lambda_0 + z$ (i.e., $\lambda$ when $z = \lambda - \lambda_0$).*

**Definition 2.3.2** (The map $F$). *The fixed-point map $F : \ell^1_\nu \to \ell^1_\nu$ is defined by*

$$F(a)_n = (a \star a)_n - c_n.$$

*A zero of $F$ corresponds to a power series squaring to $\lambda_0 + z$.*

**Definition 2.3.3** (Squaring map). *The squaring map $sq : \ell^1_\nu \to \ell^1_\nu$ is $sq(a) = a \star a$.*

**Theorem 2.3.4** (Fréchet derivative of squaring). *The squaring map is Fréchet differentiable with derivative at $a$:*

$$D_a[sq](h) = 2(a \star h).$$

*Proof.* The key identity $(a + h)^2 - a^2 - 2(a \star h) = h^2$ follows from the `CommRing` instance via the `ring` tactic. The remainder estimate $\|h^2\| \leq \|h\|^2$ uses submultiplicativity, giving the little-o condition. The `SMulCommClass` instance ensures that $2 \cdot (a \star h) = a \star (2 \cdot h)$. □

### 2.3.2 Approximate Solution

**Definition 2.3.5** (Approximate solution structure). *An approximate solution of order $N$ consists of:*

- *Finite coefficients $\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_N \in \mathbb{R}$*
- *The extension $\bar{a} \in \ell^1_\nu$ defined by $\bar{a}_n = 0$ for $n > N$*

**Definition 2.3.6** (Concrete approximate solution). *For $\lambda_0 = 1$ and $N = 2$, the approximate solution is:*

$$\bar{a}_0 = 1, \quad \bar{a}_1 = \frac{1}{2}, \quad \bar{a}_2 = -\frac{1}{8}.$$

*These are the first three Taylor coefficients of $\sqrt{1 + z}$ at $z = 0$.*

### 2.3.3 Bound Definitions

**Definition 2.3.7** ($Y_0$ bound)**.** *The $Y_0$ bound measures the initial defect:*

$$Y_0 = \|A \cdot F(\bar{a})\|_{\ell^1_\nu}$$

*where $A$ is the approximate inverse. For diagonal $A$ with entries $A_{nn} = 1/(2\bar{a}_0)$:*

$$Y_0 = \sum_{n=0}^{N} |A_{nn} \cdot F(\bar{a})_n| \nu^n + Y_0^{tail}$$

*where $Y_0^{tail}$ accounts for the tail contribution.*

**Definition 2.3.8** ($Z_0$ bound)**.** *The $Z_0$ bound measures deviation from identity:*

$$Z_0 = \|I - A \circ D_{\bar{a}} F\|.$$

*Since $D_{\bar{a}}F(h) = 2(\bar{a} \star h)$ and $A$ is diagonal with $A_{nn} = 1/(2\bar{a}_0)$:*

$$Z_0 = \sup_{n \geq 0} \left| 1 - \frac{2(\bar{a} \star e_n)_n}{2\bar{a}_0} \right| \cdot \nu^n$$

*where $e_n$ is the $n$-th standard basis sequence.*

**Definition 2.3.9** ($Z_1$ bound)**.** *The $Z_1$ bound accounts for nonlinearity:*

$$Z_1 = \frac{1}{|\bar{a}_0|} \sum_{k=1}^{N} |\bar{a}_k| \nu^k.$$

**Definition 2.3.10** ($Z_2$ bound)**.** *The $Z_2$ bound captures derivative variation:*

$$Z_2 = 2 \cdot \max\left( \|A\|, \frac{1}{2|\bar{a}_0|} \right).$$

### 2.3.4 Radii Polynomial

**Definition 2.3.11** (Radii polynomial)**.** *The radii polynomial is*

$$p(r) = Z_2 r^2 - (1 - Z_0 - Z_1) r + Y_0.$$

**Theorem 2.3.12** (Existence condition)**.** *If there exists $r_0 > 0$ such that $p(r_0) < 0$, then there exists a unique $\tilde{a} \in \ell^1_\nu$ with $\|\tilde{a} - \bar{a}\| \leq r_0$ satisfying $F(\tilde{a}) = 0$.*

### 2.3.5 Verification

**Theorem 2.3.13** (Bounds verification)**.** *For $\lambda_0 = 1/3$, $\nu = 1/4$, $N = 2$, the bounds satisfy:*

- $Y_0 < 0.001$
- $Z_0 < 0.5$
- $Z_1 < 0.3$
- $Z_2 < 4$

*These imply $p(r_0) < 0$ for some $r_0 > 0$.*

**Theorem 2.3.14** (Example 7.7 existence)**.** *For $\lambda_0 = 1/3$ and $\nu = 1/4$, there exists a unique $\tilde{a} \in \ell^1_\nu$ near $\bar{a}$ satisfying $\tilde{a} \star \tilde{a} = c$.*

### 2.3.6 Power Series Interpretation

#### Formal Power Series Connection

**Definition 2.3.15** (Embedding into PowerSeries). *The embedding* $\ell^1_\nu \hookrightarrow \texttt{PowerSeries}\ \mathbb{R}$ *sends* $a \mapsto \sum_{n=0}^{\infty} a_n X^n$.

**Definition 2.3.16** (Parameter power series). *For* $\lambda_0 \in \mathbb{R}$, *the parameter element* $c \in \ell^1_\nu$ *represents* $\lambda_0 + z$:

$$c = (\lambda_0, 1, 0, 0, \ldots) \in \ell^1_\nu.$$

**Theorem 2.3.17** (Multiplication equals Cauchy product). *For* $a, b \in \ell^1_\nu$:

$$\texttt{coeff}_n(\texttt{toPowerSeries}(a) \cdot \texttt{toPowerSeries}(b)) = (a \star b)_n.$$

*This is essentially definitional: multiplication in* $\texttt{PowerSeries}\ \mathbb{R}$ is *the Cauchy product.*

**Theorem 2.3.18** (Formal fixed-point equation). *If* $F(\tilde{a}) = 0$ *(i.e.,* $(\tilde{a} \star \tilde{a})_n = c_n$ *for all* $n$*), then at the formal level:*

$$\texttt{toPowerSeries}(\tilde{a})^2 = \texttt{paramPowerSeries}(\lambda_0).$$

#### Analytic Evaluation

**Definition 2.3.19** (Analytic evaluation). *For* $a \in \ell^1_\nu$ *and* $z \in \mathbb{R}$, *the evaluation is*

$$\texttt{eval}(a, z) := \sum_{n=0}^{\infty} a_n z^n.$$

**Lemma 2.3.20** (Absolute convergence). *If* $a \in \ell^1_\nu$ *and* $|z| \leq \nu$, *then* $\sum_{n=0}^{\infty} a_n z^n$ *converges absolutely.*

*Proof.* For $|z| \leq \nu$: $\sum_{n=0}^{\infty} |a_n z^n| \leq \sum_{n=0}^{\infty} |a_n| \nu^n = \|a\|_{\ell^1_\nu} < \infty$. □

**Theorem 2.3.21** (Mertens' theorem: evaluation commutes with multiplication). *For* $a, b \in \ell^1_\nu$ *and* $|z| \leq \nu$:

$$\texttt{eval}(a, z) \cdot \texttt{eval}(b, z) = \sum_{n=0}^{\infty} (a \star b)_n z^n.$$

*Proof.* By absolute convergence, apply Mertens' theorem: the product of absolutely convergent series equals the series of Cauchy products. □

**Theorem 2.3.22** (Squaring identity). *If* $\tilde{a} \in \ell^1_\nu$ *satisfies* $F(\tilde{a}) = 0$, *then for* $|z| \leq \nu$:

$$\texttt{eval}(\tilde{a}, z)^2 = \lambda_0 + z.$$

*Proof.* By Theorem 2.3.21: $\texttt{eval}(\tilde{a}, z)^2 = \sum_{n=0}^{\infty} (\tilde{a} \star \tilde{a})_n z^n = \sum_{n=0}^{\infty} c_n z^n = \lambda_0 + z$. □

**Corollary 2.3.23** (Evaluation identity). *For* $|\lambda - \lambda_0| \leq \nu$:

$$\texttt{eval}(\tilde{a}, \lambda - \lambda_0)^2 = \lambda.$$

### 2.3.7 Branch Selection

The equation $y^2 = \lambda$ has two solutions: $\pm\sqrt{\lambda}$. We show that the power series selects the positive branch.

**Lemma 2.3.24** (Evaluation at zero). *For any $a \in \ell^1_\nu$:*

$$\texttt{eval}(a, 0) = a_0.$$

**Theorem 2.3.25** (Positive branch identification). *Suppose:*

1. $\lambda_0 > 0$
2. $\lambda > 0$
3. $|\lambda - \lambda_0| \le \nu$
4. $\tilde{a}_0 > 0$

*Then* $\texttt{eval}(\tilde{a}, \lambda - \lambda_0) = \sqrt{\lambda}$.

*Proof.* **Step 1:** Evaluate at $z = 0$: $\texttt{eval}(\tilde{a}, 0) = \tilde{a}_0$.

**Step 2:** From $\texttt{eval}(\tilde{a}, 0)^2 = \lambda_0$ and $\tilde{a}_0 > 0$: $\tilde{a}_0 = \sqrt{\lambda_0} > 0$.

**Step 3:** Since $\texttt{eval}(\tilde{a}, \lambda - \lambda_0)^2 = \lambda > 0$, we have $\texttt{eval}(\tilde{a}, \lambda - \lambda_0) \neq 0$.

**Step 4 (Positivity by IVT):** Suppose for contradiction that $\texttt{eval}(\tilde{a}, \lambda - \lambda_0) < 0$. Since $\texttt{eval}(\tilde{a}, 0) > 0$ and $z \mapsto \texttt{eval}(\tilde{a}, z)$ is continuous (uniform convergence of power series), by the Intermediate Value Theorem there exists $c$ between 0 and $\lambda - \lambda_0$ with $\texttt{eval}(\tilde{a}, c) = 0$.

But $\texttt{eval}(\tilde{a}, c)^2 = \lambda_0 + c$, so $0 = \lambda_0 + c$, giving $c = -\lambda_0 < 0$.

*Case 1:* If $\lambda \ge \lambda_0$, then $c \in [0, \lambda - \lambda_0]$ requires $c \ge 0$, contradicting $c = -\lambda_0 < 0$.

*Case 2:* If $\lambda < \lambda_0$, then $c \in [\lambda - \lambda_0, 0]$ requires $c \ge \lambda - \lambda_0$, i.e., $-\lambda_0 \ge \lambda - \lambda_0$, so $\lambda \le 0$, contradicting $\lambda > 0$.

Therefore $\texttt{eval}(\tilde{a}, \lambda - \lambda_0) > 0$.

**Step 5:** Since $\texttt{eval}(\tilde{a}, \lambda - \lambda_0) > 0$ and $\texttt{eval}(\tilde{a}, \lambda - \lambda_0)^2 = \lambda$, we have $\texttt{eval}(\tilde{a}, \lambda - \lambda_0) = \sqrt{\lambda}$. □

### 2.3.8 Summary

**Theorem 2.3.26** (Main result for Example 7.7). *For $\lambda_0 = 1/3$, $\nu = 1/4$, and $N = 2$:*

1. ***(Existence)*** *There exists a unique $\tilde{a} \in \ell^1_\nu$ with $\|\tilde{a} - \bar{a}\|_{\ell^1_\nu} \le r_0$ satisfying $F(\tilde{a}) = \tilde{a} \star \tilde{a} - c = 0$.*
2. ***(Analytic interpretation)*** *The power series $\tilde{x}(z) = \sum_{n=0}^{\infty} \tilde{a}_n z^n$ converges for $|z| \le \nu$ and satisfies $\tilde{x}(\lambda - \lambda_0) = \sqrt{\lambda}$ for $\lambda \in [\lambda_0 - \nu, \lambda_0 + \nu]$ with $\lambda > 0$.*

*Proof.* Part (1) follows from Theorem 2.3.14 via the radii polynomial method. Part (2) follows from Theorem 2.3.25: the fixed-point equation $\tilde{a} \star \tilde{a} = c$ implies $\tilde{x}(z)^2 = \lambda_0 + z$, and the branch selection via continuity and $\tilde{a}_0 > 0$ gives the positive square root. □